Numerical Weather Prediction

Remarks on the theory and practice of potential vorticity inversion

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Abstract

Inversion of an approximate form of Ertel's potential vorticity (subject to nonlinear balance), and of quasi-geostrophic potential vorticity, have become almost routine procedures in meteorology. From the products of an inversion, a corresponding set of balanced theromodynamic and 3D flow fields can be calculated. The basis and process of potential vorticity inversion are here addressed, highlighting some of the issues involved. The properties of the two main inversion systems are briefly reviewed and contrasted. Particular attention is then paid to the choice of boundary conditions. Following Hakim *et al.* (1996), a clear distinction is drawn between prognostic inversion, for which boundary conditions are clearly defined, and static inversion, for which they are not. Whether this aspect of static inversion is advantageous or disadvantageous depends on context: in applications to field modification (in data assimilation and post-processing) it can offer useful scope, but in studies designed to develop scientific intuition it could lead to confusion. Static inversion in the presence of a mountainous lower boundary is discussed, with emphasis on the quasi-geostrophic case. An Appendix describes the formulation of a prognostic quasi-geostrophic model, including potential vorticity and omega equations, when a general vertical coordinate is used which is *not* terrain-following.

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1 Introduction

Potential vorticity (PV) inversion, as part of the apparatus of "PV thinking", has become well established as a meteorological analysis tool, following especially the work of Hoskins *et al.* (1985). In terms of the spatial form of the PV field and its temporal evolution, "PV thinking" offers a coherent view of atmospheric behaviour close to geostrophic balance. Two theoretical aspects enlighten this view: the conservation properties of PV; and the principle of invertibility, whereby (subject to appropriate balance and boundary conditions) one may obtain associated wind and thermodynamic fields from the PV field.

The precise form of PV depends on the system of equations used and the level of approximation applied (see Thorpe and Volkert (1997)). The two forms most commonly used for meteorological purposes are Ertel's PV (EPV; Ertel 1942) and quasi-geostrophic PV (QGPV; Charney and Stern 1962).

Ertel's PV is defined as

$$\mathrm{EPV} = \alpha \mathbf{Z} \cdot \nabla\theta \,. \tag{1.1}$$

Here α denotes specific volume, θ potential temperature, $\mathbf{Z}$ absolute vorticity and ∇ the 3D gradient operator. EPV is conserved on fluid parcels in frictionless, adiabatic flow. The units of EPV are $\mathrm{Km^2 s^{-1} kg^{-1}}$, and it has become common practice, following Hoskins *et al.* (1985), to define a "potential vorticity unit" (PVU) of 10^{-6} $\mathrm{Km^2 s^{-1} kg^{-1}}$. Applications of (1.1) usually incorporate the traditional and shallow-atmosphere approximations: the horizontal component of planetary vorticity is neglected in $\mathbf{Z}$, and distance from the Earth's centre is replaced by a constant mean value in all geometric factors. Hydrostatic balance is usually assumed too, when consequently the contribution of horizontal variations of vertical velocity to $\mathbf{Z}$ is neglected. See White and Bromley (1995) for further discussion.

With pressure p as vertical coordinate, QGPV is defined as

$$\mathrm{QGPV} = \nabla_p^2 \Psi + f + \frac{\partial}{\partial p}\left(\frac{f_0^2}{\sigma^2}\frac{\partial \psi}{\partial p}\right), \tag{1.2}$$

where ψ denotes the geostrophic streamfunction, ∇_p the 2D gradient operator on surfaces of constant pressure, f_0 a mean value of the Coriolis parameter f, and σ^2 the static stability of a horizontally uniform reference atmosphere (see (5.10), below). In frictionless, adiabatic motion, QGPV is conserved following the geostrophic flow. Its units are $\mathrm{s^{-1}}$. QGPV as defined by (1.2) is not an approximation to EPV as defined by (1.1); rather, the conservation of QGPV following the (horizontal) geostrophic flow is an approximate representation of the conservation of EPV following the full 3D flow (Bretherton 1966, p328). For this reason, QGPV is sometimes referred to as the QG pseudo potential vorticity (Charney 1971). [In the context of the shallow water equations, which we shall not discuss again in this paper, QGPV *is* a direct approximation to the corresponding EPV - to within a constant factor having the dimensions of length - because shallow-water flow is essentially 2-dimensional.]

Inversion of a PV field involves the assumption of a diagnostic approximation to the horizontal divergence equation, often referred to simply as a *balance condition*; see Hoskins *et al.* (1985). To invert an EPV field we must stipulate the balance relationship - various choices are possible, as we shall discuss. When inverting a QGPV field, however, there is no scope for choice of balance condition, as geostrophic balance is implicit in the definition (1.2) of QGPV. It is, of course, always necessary to specify appropriate boundary conditions for the inversion, whether of EPV or of QGPV.

A system that inverts an approximation to EPV subject to nonlinear balance (Charney 1955) was devised by Davis and Emanuel (1991) (hereafter DE) and has been widely used; see Demirtas and Thorpe (1999), Korner and Martin (2000) and Swarbrick (2001). The QGPV inversion system has been widely used too; see Hakim *et al.* (1996) (hereafter HKB) and Carroll (1997). Inversion systems based on semi-geostrophy (Birkett and Thorpe 1997) and on asymmetric balance (Moller and Jones 1998) have also been used. It is also possible to choose a linear form of the balance equation, with full latitude variation of the Coriolis parameter, in conjunction with EPV. (Replacing f_0 by f in the QGPV (1.2) is possible too, but might be considered an inconsistent procedure according to familiar rules for formulating prognostic QG models.) In this paper we focus on EPV inversion subject to nonlinear balance ("the EPV system") and on QGPV inversion ("the QGPV system"). We shall allude briefly to variants of the EPV system in which different balance conditions are stipulated.

Whether one chooses to invert the EPV system or the QGPV system, several issues have to be confronted. As commonly formulated, both the QGPV and EPV systems employ pressure as the vertical coordinate, either directly (for QGPV inversion) or via the Exner coordinate form (for EPV inversion - see section 2). The lower limit of the atmosphere is not then a coordinate surface. This problem may be avoided, if not overcome, by terminating the inversion domain on a pressure surface which does not intersect the Earth. (The same situation arises when height (z) above mean sea level, or indeed any quantity that is not constant at the terrain height, is used as vertical coordinate.)

A more fundamental issue is the nature of the boundary conditions themselves. In most practical situations one wishes to invert PV over a limited (i.e. non-global) horizontal region. Thus, as well as conditions to be applied on quasi-horizontal surfaces, conditions to be applied on the vertical boundaries defining the limits of the horizontal domain must be chosen. HKB discuss the choice of boundary conditions on horizontal surfaces in the QGPV case, and Shapiro (1996) considers the effect of different boundary conditions on vertical surfaces in the EPV inversion system.

Both the QGPV and EPV systems approximate the winds to some extent. Inversion of QGPV returns a non-divergent geostrophic wind, while inversion of the EPV system delivers a non-divergent wind defined by the nonlinear balance equation. In either system, one can derive implied fields of vertical motion and divergent wind via appropriate "omega" and continuity equations (Thorpe 1997, Ziemianski and Thorpe 2000).

In sections 2 and 3, we briefly describe the properties of the EPV and QGPV inversion systems. Boundary conditions are discussed in section 4, paying particular attention to the distinction between *prognostic* and *static* PV inversion, as defined by HKB, in relation to the choice of horizontal boundary conditions. In sections 5 and 6 we discuss the process of static QGPV inversion when the lower boundary condition is mountainous. The Appendix describes the formulation of a prognostic QG model which is appropriate for a general vertical coordinate which is not terrain-following.

2 Properties of the EPV inversion system

DE develop their approximate EPV inversion system by making the Helmholtz decomposition of the wind into rotational and divergent parts and then neglecting the latter in comparison with the former. The adopted vertical coordinate is Exner pressure, $\pi = (p/p_{ref})^{\kappa}$, where $\kappa \equiv R/c_p$ and $p_{ref} = 1000$hPa, and the horizontal coordinates are longitude λ and latitude φ. Applying DE's procedure to (1.1) leads to the following approximate expression for EPV:

$$\mathrm{EPV} = \frac{g\kappa\pi}{p}\left\{\left(f + \nabla_\pi^2\Psi\right)\frac{\partial^2\Phi}{\partial\pi^2} - \frac{1}{a^2\cos^2\varphi}\frac{\partial^2\Psi}{\partial\lambda\partial\pi}\frac{\partial^2\Phi}{\partial\lambda\partial\pi} - \frac{1}{a^2}\frac{\partial^2\Psi}{\partial\varphi\partial\pi}\frac{\partial^2\Phi}{\partial\varphi\partial\pi}\right\}. \qquad (2.1)$$

Here a is the mean radius of the Earth, Ψ is the streamfunction of the rotational wind, and Φ is the geopotential.

The accompanying balance equation is that suggested by Charney (1955). The form of this equation given by DE omits a number of terms that arise as a consequence of the use of a spherical polar coordinate system. Houghton (1968) gives the full form for spherical polar

coordinates; in π-coordinate form it is

$$\nabla_\pi^2\Phi = f\nabla_\pi^2\Psi + \nabla_\pi\Psi\cdot\nabla_\pi f + \frac{1}{a^4\cos^2\varphi}\Big\{2\Psi_{\lambda\lambda}\Psi_{\varphi\varphi} - 2\Psi_{\lambda\varphi}^2$$

$$-4\Psi_\lambda\Psi_{\lambda\varphi}\tan\varphi - \Psi_\lambda^2\left(1+2\tan^2\varphi\right)\Big\} - \frac{\Psi_\varphi^2}{a^4} - \frac{2\tan\varphi}{a^4}\Psi_\varphi\Psi_{\varphi\varphi}\,. \qquad (2.2)$$

The terms DE omit are those on the second line of (2.2). The nonlinear balance/EPV system defined by (2.1) and the Charney (1955) balance equation may be inverted for Ψ and Φ (given EPV) if suitable boundary conditions are specified. Similar remarks apply to the system defined by (2.1) and the full spherical polar, nonlinear balance equation (2.2).

The nonlinearity of the EPV inversion system is the source of a number of difficulties. Convergence in iterative numerical solution may be problematic. Using a method involving successive over-relaxation, DE found it necessary to constrain EPV to be positive everywhere in order to obtain convergence; any negative values were set to a small positive constant.

As the EPV inversion system consists of a pair of coupled nonlinear equations, it is not obvious that any solution of the system will be unique (for a given set of boundary conditions). DE state that the solution they obtained is either unique or that any other solutions that may exist are unattainable using their technique.

Another difficulty concerns the *attribution* (Bishop and Thorpe 1994) of particular features of the solution to particular localised features, or "anomalies", of the EPV field. Attribution is a key feature of the use of potential vorticity inversion to increase insight into atmospheric behaviour near geostrophic balance. [A precise definition of the anomaly of a QGPV field is given below in (4.5). For the purposes of the current discussion, the anomaly of an EPV field may be regarded as its deviation from the variation due to that of the planetary vorticity f and of a horizontally-averaged static stability.] As the EPV inversion system is nonlinear, the linear superposition of results obtained for spatial decompositions of the EPV distribution could be misleading. One may linearise the equations, which introduces ambiguity into the inversion process as there are a number of ways in which the linearisation may be carried out. Davis (1992) presents various methods for carrying out such "piecewise inversions" and puts an upper bound of 15-25% (in the vicinity of the EPV anomaly) on the difference between them. Thorsteinsson *et al.* (1999) have also shown that any residual errors due to the non-linearity are small for a particular case. Using a semi-geostrophic PV (SGPV) system, which is also nonlinear, Birkett and Thorpe (1997) have shown that provided the SGPV anomalies are well separated the responses to them are quasi-linearly superposable.

The EPV inversion system is not equivalent to any of the energetically-consistent prognostic models identified in Lorenz's (1960) classical study. This inevitably arouses misgivings. Other unsettling features are the freedom (a) to choose a balance relation, (b) to approximate the balance relation (as noted above regarding (2.2)), and (c) to partially linearise the whole system (as noted in connection with "piecewise inversions"). We should note, however, that in recent years balanced *prognostic* models based on PV conservation rather than energy conservation have been constructed and successfully operated by several workers; see McIntyre and Norton (2000) and references therein.

3 Properties of the QGPV inversion system

The QGPV system suffers from the disadvantage that it depends on more severe scaling assumptions than does the EPV system. Nevertheless, it is useful for the investigation of a wide range of extratropical atmospheric phenomena. Unlike the EPV system, the QGPV system is directly related to a classical prognostic formulation which has good conservation properties: the quasi-geostrophic model (Charney and Stern 1962).

QGPV (see (1.2)) involves a single dependent variable, ψ, and does not require a separate balance equation. It has the form of a linear elliptic differential operator acting on ψ, so there are few problems with convergence of iterative methods of inversion, and no possibility of non-uniqueness of the solution (subject to stated boundary conditions).

The linearity of QGPV has another important consequence. One can directly invert individual QGPV anomalies and linearly superpose the results without further approximation. So QGPV inversion allows unambiguous attribution of atmospheric features to individual QGPV anomalies (for a given set of boundary conditions) at the price of less accuracy than EPV inversion. If one is primarily interested in the synoptic and larger scale flow (characterised by small Rossby number), QGPV inversion is an attractive alternative to EPV inversion, and several studies have used it; see Robinson (1988) and Henderson *et al.* (1999) as well as HKB and Carroll (1997). However, if one is interested in mesoscale flow, the work of Davis (1992) suggests that even given the ambiguities in a piecewise inversion of EPV it is to be preferred to the use of QGPV for accurate work: the differences between the results of EPV inversions using various linearisations are considerably smaller than the differences between the EPV results and the QGPV result.

4 Choice of boundary conditions

The question of the boundary conditions on vertical surfaces is relatively straightforward. The geopotential perturbation associated with an isolated PV anomaly is reasonably expected to become small at large horizontal distances from the anomaly. This condition may be quite well approximated by employing homogeneous Dirichlet boundary conditions on the vertical boundaries (whereby the geopotential *perturbation*, i.e. the geopotential response to the anomaly, is specified as zero) provided that these boundaries are sufficiently far away from the PV anomaly. DE employ this strategy to define the boundary conditions on vertical surfaces in their EPV inversion system. The analysed geopotential is used to specify Φ on the vertical boundaries, and the boundary values of the streamfunction, Ψ, are obtained from the analysed winds, $\mathbf{v}$, using the expression

$$\frac{\partial \Psi}{\partial s} = -\mathbf{v} \cdot \mathbf{n} + \frac{\oint \mathbf{v} \cdot \mathbf{n} ds}{\oint ds} \,, \tag{4.1}$$

in which $\mathbf{n}$ is the outward unit normal on the boundary and s is parallel to the edge of the horizontal area enclosed. This form ensures that $\partial\Psi/\partial s$ vanishes when integrated horizontally around the boundary. Shapiro (1996) found that the results of an EPV inversion were insensitive to the choice of vertical boundary conditions, provided that the boundaries were sufficiently far removed from the EPV anomaly being inverted.

The question of the choice of horizontal boundary conditions is more critical. In our discussion we shall use perhaps the simplest 3-dimensional formulation of QGPV. The quasi-geostrophic, frictionless adiabatic motion of an incompressible fluid on a mid-latitude β-plane is governed by the vorticity and thermodynamic equations in the following nonlinear forms:

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_g \cdot \nabla_z\right) \left\{\nabla_z^2 \psi + \beta y\right\} = f_0 \frac{\partial w}{\partial z} \,, \tag{4.2}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_g \cdot \nabla_z\right) \frac{\partial \psi}{\partial z} + \frac{N^2}{f_0} w = 0 \,. \tag{4.3}$$

Here $\mathbf{v}_g = \mathbf{k} \times \nabla_z \psi$ is the geostrophic flow ($\mathbf{k}$ being unit vector in the upward vertical), ∇_z is the horizontal gradient operator $(\partial/\partial x \,, \partial/\partial y)$ on constant height (z) surfaces, and ψ is the streamfunction of $\mathbf{v}_g$; t is time, w is vertical velocity component, and N is the reference state buoyancy frequency (a function of z only). The Coriolis parameter f is represented in (4.3) as $f = f_0 + \beta y$, where f_0 and β are constants and y is the meridional coordinate.

Eliminating w between (4.2) and (4.3) leads to

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_g \cdot \nabla_z\right) \left\{\nabla_z^2 \psi + \beta y + \frac{f_0^2}{N^2} \frac{\partial^2 \psi}{\partial z^2}\right\} = 0 \,. \tag{4.4}$$

The quantity in curly brackets in (4.4) is this model's QGPV, and the QGPV *anomaly* is

$$\text{QGPV} - \beta y \equiv \nabla_z^2 \psi + \frac{f_0^2}{N^2} \frac{\partial^2 \psi}{\partial z^2} \,. \tag{4.5}$$

For simplicity, we have assumed that the buoyancy frequency N is independent of height as well as horizontal location and time. The conclusions drawn in this section can all be generalised to the case in which N varies with height, and indeed to the case of a compressible fluid - when a reference state density profile appears in (4.2), (4.4) and (4.5) (see (6.3)).

We apply standard β-plane channel conditions on the prognostic system (4.2)-(4.4):

$$w = 0 \text{ on } z = 0,\, H \,; \tag{4.6}$$

$$\psi^* \equiv \psi - [\psi] = 0 \text{ on } y = 0,\, L \,; \tag{4.7}$$

$$[\partial \psi / \partial y] = 0 \text{ on } y = 0,\, L \,. \tag{4.8}$$

In (4.7) and (4.8), [...] indicates the zonal (x) average of the operand. Condition (4.6) represents rigid horizontal boundaries at $z = 0$ and $z = H$, (4.7) ensures no geostrophic flow through $y = 0$ and $y = L$, and (4.8) ensures conservation of energy and angular momentum.

A diagnostic equation for w results when the local time derivative terms are eliminated between (4.2) and (4.3):

$$\frac{N^2}{f_0} \nabla_z^2 w + f_0 \frac{\partial^2 w}{\partial z^2} = G \,. \tag{4.9}$$

The source term G in (4.9) is a well-defined function of ψ and its spatial derivatives. G can be written in many different forms, but this aspect is not relevant here; the important point is that if ψ is known, then the Poisson equation (4.9) can be solved for w using (4.6) and appropriate lateral conditions (obtainable from (4.2), (4.3) and (4.6) - (4.8)).

Equation (4.4) is a Poisson equation for the *geostrophic streamfunction tendency* $\partial\psi/\partial t$:

$$\left(\nabla_z^2 + \frac{f_0^2}{N^2}\frac{\partial^2}{\partial z^2}\right)\frac{\partial\psi}{\partial t} = -\left(\mathbf{v}_g \cdot \nabla_z\right)\mathrm{QGPV} \equiv F_Q \,. \tag{4.10}$$

Boundary conditions on $\partial\psi/\partial t$ at $z = 0,\ H$ are obtained from (4.3) by applying (4.6):

$$\frac{\partial}{\partial z}\left(\frac{\partial\psi}{\partial t}\right) = -\left(\mathbf{v}_g \cdot \nabla_z\right)\frac{\partial\psi}{\partial z} \equiv F_T \ \text{ at } \ z = 0,\ H. \tag{4.11}$$

The conditions (4.11) are of inhomogeneous Neumann form.

The Poisson equation (4.10) may be inverted for $\partial\psi/\partial t$, given (4.11) and ψ (from which QGPV and hence F_Q may be evaluated). Discretized versions of this procedure are standard in the time integration of quasi-geostrophic models; see, for example, Holton (1992).

Inverting (4.10) for $\partial\psi/\partial t$ given (4.11) has been aptly termed *prognostic* PV inversion by HKB, who distinguish it from *static* PV inversion. In static PV inversion, one specifies QGPV as a function of space, and inverts (4.5) as a Poisson equation for the streamfunction ψ. HKB (see also Thorpe (1997), p2380) note that prognostic and static PV inversion differ in a key respect: the boundary conditions in prognostic PV inversion are clearly defined (by (4.11)), but there is no physical principle which imposes a particular choice of boundary conditions in static PV inversion (though some choices are excluded - see below). Static PV inversion is a specific term for what is often referred to simply as "PV inversion".

It might be suspected that the requirement that w vanish on rigid horizontal boundaries (i.e. condition (4.6)) would be violated unless a particular choice of conditions were to be applied in static PV inversion. That this is not the case may be understood by noting that the diagnostic equation (4.9) determines w for any given source function G (the boundary conditions being supplied by (4.2), (4.3) and (4.6) - (4.8)). Via (4.11), condition (4.6) constrains only the streamfunction *tendency*. Again, it might be suspected that static PV inversion subject to non-Neumann conditions at $z = 0,\ H$ would be somehow inconsistent with a subsequent time integration subject to the inhomogeneous Neumann conditions (4.11) that are necessary in the prognostic case. To see that this is not the case either, note that given any (suitably differentiable) field of ψ, (i) the QGPV field can be evaluated from the definition (4.5), (ii) the source function F_Q in (4.10) can be determined, and (iii) a time integration can be started subject to the necessary prognostic Neumann conditions (4.11). Time integration of a quasi-geostrophic model puts no restriction on the initial field of ψ.

These conclusions are unchanged if one applies the horizontal boundary conditions using the generalised QGPV scheme of Bretherton (1966). The Bretherton scheme is a convenient way of homogenising Neumann boundary conditions by extending the definition of QGPV, but it does not constitute a change of boundary condition type.

As HKB note, not all choices of boundary conditions are allowable in static PV inversion. Eq. (4.5) may be integrated over the whole domain to give

$$\int\int\int \left(\mathrm{QGPV} - \beta y\right) dxdydZ = \int\int \left(\nabla\psi\right)\cdot\mathbf{n}dS \,. \tag{4.12}$$

Here $\mathbf{n}$ is unit outward normal on the boundaries of the domain and dS is the surface element; $Z = Nz/f_0$ and ∇ is the 3D gradient operator in $x,\ y,\ Z$ space. If Neumann conditions are

applied on *all* boundaries (an admittedly extreme choice), the *regularity condition* (4.12) will be violated unless QGPV = QGPV $(x,\ y,\ z)$ is appropriately specified. Dirichlet conditions do not suffer from this restriction, since - given any distribution of QGPV - normal gradients of ψ on the boundaries can adjust so that (4.12) is obeyed.

The regularity condition (4.12) and the possibility of violating it may easily be interpreted. The problem of inverting (4.5) for ψ, given (QGPV $-\ \beta y$), is mathematically isomorphic to the physically different problem of determining the steady-state *temperature* distribution, $T(\mathbf{r})$, in a *solid* of unit thermal diffusivity, specific heat and density, given a specified *heat source* Q:

$$\frac{\partial T}{\partial t} = 0 = \nabla^2 T + Q\ . \tag{4.13}$$

If we supply heat within a volume V of the solid, and we insist that the normal temperature gradient at the bounding surface S should take certain values (i.e. we impose Neumann conditions) then we should expect to find no steady solution unless the total rate of heat supply to the interior happens to balance the total heat flux through the boundaries:

$$-\int\int_S \frac{\partial T}{\partial n} dS = \int\int\int_V Q dV\ . \tag{4.14}$$

On the other hand, if we insist that the temperature at the bounding surface should take certain values (i.e. if we impose Dirichlet conditions) then a steady-state solution will exist for any Q because the normal gradients of T on S can adjust so as to satisfy (4.14).

Perhaps the simplest case of this thought experiment in heat diffusion is the one in which the solid is thermally insulated from its surroundings. Obviously, in this case no steady-state temperature distribution will occur in the solid unless the *net* heating vanishes; the boundary conditions are homogeneous Neumann, and (from (4.14)) regularity of the steady-state problem requires the integral of Q over the volume to vanish .

Amongst the many other physical contexts that yield Poisson's equation, and hence offer analogues for static or prognostic PV inversion, electrostatics is the most widely exploited to date. Bishop and Thorpe (1994) provide a useful physical picture of the consequences of imposing Neumann and Dirichlet boundary conditions in terms of the electrostatics analogy. They show that applying a homogeneous Neumann boundary condition may be visualized in terms of an "image" QGPV anomaly of the same sign as the actual one, with the boundary acting as an "insulator". The application of a homogeneous Dirichlet boundary condition is visualized in terms of an opposite sign image QGPV anomaly with a "conducting" boundary. [This terminology is virtually unaltered if one has in mind the steady-state heat conduction problem discussed in the previous paragraph.]

Perhaps partly because inhomogeneous Neumann boundary conditions are required in prognostic PV inversion, they have frequently been applied in static PV inversion also. For example, in their EPV system, DE apply Neumann conditions on the horizontal boundaries, using the analysed potential temperature, θ, to specify $\partial\Phi/\partial\pi$ via hydrostatic balance. The corresponding boundary conditions for Ψ are obtained from those for Φ by assuming geostrophy, which DE justify by scaling arguments.

If condition (4.12) is obeyed by the distribution of the anomaly (QGPV $-\ \beta y$), static QGPV inversion is a consistently-posed problem even if Neumann conditions are imposed on all boundaries. Static QGPV inversion is consistently posed, whatever the distribution

of (QGPV $-\beta y$), if Dirichlet conditions are applied on some boundaries and Neumann conditions on others. However, in such cases - of which a common example is the application of Dirichlet conditions on vertical boundaries and Neumann conditions on quasi-horizontal boundaries - it is clear that the Neumann boundaries are playing a constraining role, while gradients at the Dirichlet boundaries are playing an adjusting role (so that (4.12) is satisfied). This situation does not bode well for realistic solutions if the Dirichlet boundaries are remote from the anomaly and are thus considered to be in regions of small response to it.

The point at issue is the status of the many choices of boundary conditions that do imply a consistently-posed problem. Two general observations may be made. First, if one has accurate information about both ψ and its normal gradient at the boundaries, then the result of inverting the corresponding (accurately determined) QGPV will be the same whether one specifies Dirichlet conditions using the information about ψ or Neumann conditions using the information about its normal gradient. Second, the difference δ between the solutions of two Poisson problems that differ only with respect to boundary conditions (in the same domain!) must obey $\nabla^2\delta = 0$ in the interior; by a well-known property of Laplace's equation, the extrema of δ must therefore occur somewhere on the boundary. Differences between QGPV inversions carried out using different boundary conditions will thus be largest somewhere on the boundaries themselves, rather than in the interior.

HKB tested three options for the horizontal boundary conditions: homogeneous Dirichlet, homogeneous Neumann, and "inhomogeneous" Neumann - whereby the perturbation potential temperature on the boundary is specified as zero everywhere except in the regions enclosed by the area of the QGPV anomaly. Inhomogeneous Dirichlet conditions were also explored. In all cases, homogeneous Dirichlet conditions were applied on vertical boundaries (so that the regularity condition (4.12) was obeyed). After a comparison of inversions using each set of boundary conditions and the free-space Green's function solutions of Bishop and Thorpe (1994) (for which decay at infinity is required), HKB adopted a homogeneous Dirichlet condition on horizontal boundaries for reasons which we paraphrase as follows:

1. Contributions to the r.h.s. of constraint (4.12) are nowhere compromised by the boundary conditions.
2. The solutions are less sensitive to changes in the horizontal boundary conditions from homogeneous to inhomogeneous form than in the Neumann case.
3. The three-dimensional structure of the wind and potential temperature perturbation fields recovered from the inversion in the case of an upper-level PV anomaly are more consistent with qualitative expectation.
4. The solution compares well with the free-space Green's function solution.

Thus, if one has no information about the horizontal boundaries, homogeneous Dirichlet conditions are recommended. However, one may have information that suggests a different choice. An example of this situation is provided by Huo *et al.* (1998), where a surface temperature anomaly is used as a surrogate EPV anomaly. In these circumstances it is natural to apply an inhomogeneous Neumann condition on the lower boundary via the hydrostatic relation, as θ is known at the surface *a priori.* Clearly, the choice of horizontal boundary condition will depend partly on the available information.

For both definiteness and clarity, our discussion has concentrated on QGPV inversion

(though we noted two EPV studies en route). Due to its nonlinear nature, the question of horizontal boundary conditions for EPV inversion is less easily analysed. However, the essence of the above discussion will apply to the inversion of EPV anomalies: the boundary conditions to be applied in (static) EPV inversion are not clearly defined, and results will depend on the choices made. Ziemianski and Thorpe (2000) find that the results of EPV inversions may change noticeably when Neumann conditions are replaced by Dirichlet conditions, but they consider that qualitative interpretation is unaffected. Griffiths *et al.* (2000) also discuss the use of Dirichlet and Neumann conditions in EPV inversion.

5 QG dynamics in a general vertical coordinate system

We next consider static QGPV inversion in the presence of mountains. First we present a prognostic QG model in a general vertical coordinate η that varies monotonically with height - a derivation is given in the Appendix. Our aim is to provide for QG theory a general formulation comparable to that of Kasahara (1974) for the hydrostatic primitive equations; but it turns out that our model is not appropriate for terrain-following coordinates. Ways around this restriction are considered in section 6.

Kasahara's (1974) equations (in frictionless, adiabatic form) are our starting point, and we introduce a reference state of rest and hydrostatic balance in which all fields are functions only of the vertical coordinate η. The reference state is denoted by the subscript 0, and the deviation from it by a prime. For example,

$$\alpha = \alpha_0(\eta) + \alpha'(x,\, y,\, \eta,\, t)$$

represents the specific volume α, and similar expressions represent pressure p, geopotential Φ and potential temperature θ as deviations p', Φ' and θ' from the reference profiles $p_0(\eta)$, $\Phi(\eta)$ and $\theta(\eta)$. We define the geostrophic flow $\mathbf{v}_g$ and an associated streamfunction ψ by

$$\mathbf{v}_g = \frac{1}{f_0}\mathbf{k} \times \nabla_\eta \left(\Phi' + \alpha_0 p'\right) = \mathbf{k} \times \nabla_\eta \psi. \tag{5.1}$$

It is convenient also to define: (i) an extended ageostrophic flow $\widehat{\mathbf{v}}_{ag}$ as

$$\widehat{\mathbf{v}}_{ag} = \mathbf{v} - \mathbf{v}_g - \frac{\alpha'}{f_0}\mathbf{k} \times \nabla_\eta p'\,; \tag{5.2}$$

(ii) an extended vertical velocity $\widehat{w}$ by

$$\widehat{w}\frac{dp_0}{d\eta} = \dot{\eta}\frac{dp_0}{d\eta} + \left(\frac{\partial}{\partial t} + \mathbf{v}_g \cdot \nabla_\eta\right) p'\,; \tag{5.3}$$

and (iii) a reference-state static stability factor S_0 as

$$S_0 = S_0(\eta) \equiv \frac{\alpha_0}{\theta_0}\frac{d\theta_0}{d\eta}\,. \tag{5.4}$$

From these definitions, the procedure given in the Appendix leads to

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_g \cdot \nabla_\eta\right)\left\{\nabla_\eta^2\psi + \beta y - f_0^2\left(\frac{dp_0}{d\eta}\right)^{-1}\frac{\partial}{\partial \eta}\left(\frac{1}{S_0}\frac{\partial \psi}{\partial \eta}\right)\right\} = 0\,, \tag{5.5}$$

which is the η-coordinate conservation equation for QGPV, and

$$f_0 \frac{\partial}{\partial \eta} \left\{ \left(\frac{dp_0}{d\eta} \right)^{-1} \frac{\partial}{\partial \eta} \left(\widehat{w} \frac{dp_0}{d\eta} \right) \right\} - \frac{S_0}{f_0} \frac{dp_0}{d\eta} \nabla_\eta^2 \widehat{w} = -2 \nabla_\eta \cdot \mathbf{Q} + \beta \frac{\partial v_g}{\partial \eta} \,, \tag{5.6}$$

which is the η-coordinate QG omega equation. From (5.3), $\widehat{w} dp_0/d\eta$ is a QG approximation to the pressure coordinate "vertical velocity" $\omega \equiv Dp/Dt$; see (A.32) and (A.33). The η-coordinate Q-vector in (5.6) is defined as:

$$\mathbf{Q} = (Q_x,\, Q_y) = \left(\frac{\partial \mathbf{v}_g}{\partial x} \cdot \nabla_\eta \frac{\partial \psi}{\partial \eta} \,,\; \frac{\partial \mathbf{v}_g}{\partial y} \cdot \nabla_\eta \frac{\partial \psi}{\partial \eta} \right) \,. \tag{5.7}$$

If we set $\eta = p$ in the quantity in curly brackets in (5.5), and in (5.6), we recover the familiar pressure coordinate forms of QGPV and the omega equation respectively:

$$\text{QGPV} = \nabla_p^2 \psi + \beta y - f_0^2 \frac{\partial}{\partial p} \left(\frac{1}{S_0(p)} \frac{\partial \psi}{\partial p} \right) \,, \tag{5.8}$$

$$f_0 \frac{\partial^2 \widehat{w}}{\partial p^2} - \frac{S_0(p)}{f_0} \nabla_p^2 \widehat{w} = -2 \nabla_p \cdot \mathbf{Q} + \beta \frac{\partial v_g}{\partial p} \,. \tag{5.9}$$

Equation (5.8) is identical to (1.2), with

$$\sigma^2 = \sigma^2(p) = -S_0(p) \equiv -\frac{\alpha_0(p)}{\theta_0(p)} \frac{d\theta_0}{dp} \,. \tag{5.10}$$

The extended vertical velocity $\widehat{w}$ reduces *precisely* to $\omega \equiv Dp/Dt \equiv \dot{p}$ in this case, since $p' = 0$ and $\eta = p \,(= p_0)$. Similar substitutions may be made for other choices of the vertical coordinate. For example, setting $\eta = \theta$ in the advected quantity in (5.5) delivers the isentropic-coordinate QGPV of Berrisford *et al.* (1993):

$$\text{QGPV} = \nabla_\theta^2 \psi + \beta y - f_0^2 \left(\frac{dp_0}{d\theta} \right)^{-1} \frac{\partial}{\partial \theta} \left(\frac{\theta}{\alpha_0(\theta)} \frac{\partial \psi}{\partial \theta} \right) \,. \tag{5.11}$$

This η-coordinate QG formulation has a major shortcoming, however: *it is not suitable for terrain-following vertical coordinates* (as used in most current NWP models). The reference state for the geopotential (gz) has been chosen as $\Phi_0(\eta)$, which means that height z is constant on η surfaces in the reference state. Consider a case in which a mountain is present but there is no flow. As the reference state has Φ_0 constant on η surfaces, such a no-flow case can only be described in our formulation by having non-zero Φ'. (Consequently, other primed quantities will also be non-zero.) The condition of no flow will be described by a geostrophic flow and an equal and opposite ageostrophic flow. Not only is this inelegant, but the formulation has been derived by treating the geostrophic flow and the ageostrophic flow differently, under the assumption that the former is much larger than the latter (see Appendix). So our formulation of QGPV is appropriate only for non-terrain-following coordinates (such as height, pressure, density or potential temperature). It is worth noting, however, that the formulation *would be* suitable for surface-normalized coordinates (such as $p/p_{surface}$) in the *absence* of orography.

6 Static QGPV inversion in the presence of mountains

One way of formulating static QGPV inversion in the presence of mountains is to transform the pressure (or height or isentropic) coordinate form of QGPV directly to a terrain-following system. We first derive from the pressure-coordinate QGPV, given by (1.2), the corresponding form in terms of a pressure-based, terrain-following vertical coordinate η. The pressure-coordinate definition of streamfunction in terms of geopotential, $\psi = \Phi/f_0$, is retained, and the static stability parameter σ^2 (see (5.10)) remains a function only of *pressure*. We work in the spherical polar systems $(\lambda,\, \varphi,\, p)$ and $(\lambda,\, \varphi,\, \eta)$.

The main task is to transform $\nabla_p^2\psi$. In the $(\lambda,\, \varphi,\, p)$ system,

$$\nabla_p^2\psi = \frac{1}{a^2\cos^2\varphi}\left\{\left.\frac{\partial^2\psi}{\partial\lambda^2}\right|_p + \cos\varphi\,\frac{\partial}{\partial\varphi}\bigg|_p\left(\cos\varphi\,\left.\frac{\partial\psi}{\partial\varphi}\right|_p\right)\right\}\,. \tag{6.1}$$

By applying standard transformations to convert the differential operators from pressure-coordinate to η-coordinate forms, we arrive at the following expression for the pressure coordinate form of QGPV, (1.2), written in terms of η:

$$\begin{aligned}\mathrm{QGPV} = \nabla_\eta^2\psi + f + f_0^2\frac{\partial\eta}{\partial p}\frac{\partial}{\partial\eta}\left(\frac{1}{\sigma^2(p)}\frac{\partial\eta}{\partial p}\frac{\partial\psi}{\partial\eta}\right) - \frac{\partial\eta}{\partial p}\frac{\partial\psi}{\partial\eta}\nabla_\eta^2 p \\ -2\frac{\partial\eta}{\partial p}\left(\nabla_\eta p\cdot\nabla_\eta\frac{\partial\psi}{\partial\eta}\right) + \frac{\partial\eta}{\partial p}\nabla_\eta p\cdot\left\{\frac{\partial}{\partial\eta}\left(\frac{\partial\eta}{\partial p}\frac{\partial\psi}{\partial\eta}\nabla_\eta p\right)\right\}\,.\end{aligned} \tag{6.2}$$

This lengthy expression remains tractable to standard finite-difference discretization. It involves both p and η, but the definition $\eta = \eta(p,\, p_s)$ of the pressure-based terrain-following coordinate enables p to be eliminated in favour of the surface pressure, $p_s(x,\, y)$.

The same method may be applied to transform the height-coordinate form of QGPV to a height-based terrain-following coordinate. Our starting point is the expression for QGPV using height (z) as vertical coordinate (see, for example, White (1977)):

$$\mathrm{QGPV} = \nabla_z^2\psi + f + \frac{f_0^2}{\rho_0(z)}\frac{\partial}{\partial z}\left(\frac{\rho_0(z)}{N^2(z)}\frac{\partial\psi}{\partial z}\right)\,. \tag{6.3}$$

Here $\rho_0(z)$ is the reference state density, and (in terms of the corresponding $\theta_0(z)$)

$$N^2(z) = \frac{g}{\theta_0(z)}\frac{d\theta_0}{dz}\,. \tag{6.4}$$

The streamfunction $\psi = p^{'}/\left(\rho_0(z)f_0\right) = \left(p - p(z_0)\right)/\left(\rho_0(z)f_0\right)$. By applying standard transformations, we arrive at an expression for the height coordinate form of QGPV written in terms of a height-based terrain-following vertical coordinate, which we again denote by η:

$$\begin{aligned}\mathrm{QGPV} = \nabla_\eta^2\psi + f + \frac{f_0^2}{\rho_0(z)}\frac{\partial\eta}{\partial z}\frac{\partial}{\partial\eta}\left(\frac{\rho_0(z)}{N^2(z)}\frac{\partial\eta}{\partial z}\frac{\partial\psi}{\partial\eta}\right) - \frac{\partial\eta}{\partial z}\frac{\partial\psi}{\partial\eta}\nabla_\eta^2 z \\ -2\frac{\partial\eta}{\partial z}\left(\nabla_\eta z\cdot\nabla_\eta\frac{\partial\psi}{\partial\eta}\right) + \frac{\partial\eta}{\partial z}\nabla_\eta z\cdot\left\{\frac{\partial}{\partial\eta}\left(\frac{\partial\eta}{\partial z}\frac{\partial\psi}{\partial\eta}\nabla_\eta z\right)\right\}\,.\end{aligned} \tag{6.5}$$

Both z and η are involved here, but the definition $\eta = \eta(z, z_s)$ of the terrain-following coordinate allows z to be eliminated in favour of the orographic height, $z_s(x, y)$. This is more convenient than in the pressure-based case, since $z_s(x, y)$ is a known function.

In the case either of (6.3) or of (6.5), inversion of the relevant QGPV requires a specification of boundary conditions. In the terrain-following system, the lower boundary condition can be applied at the actual height (or pressure) of the terrain itself; indeed that is the main point of working in the terrain-following system. It should be noted, however, that this procedure for static QGPV inversion cannot be identified with a consistent *prognostic* quasi-geostrophic system. To preserve the conservation properties of a prognostic QG model it is necessary to represent the effects of mountainous terrain (in terms of the resulting vertical velocity) at the lowest coordinate surface of the model rather than at the actual height of the terrain. One could of course carry out static QGPV inversion in the pressure- or height-coordinate system, and apply the chosen boundary conditions at the lowest coordinate surface of the model rather than at the actual terrain height. This approach - which is familiar from classical linearised studies - would make the use of a terrain-following coordinate unnecessary, but would be acceptably accurate so long as the variation in terrain height across the domain were small.

7 Conclusion

We set out in this paper to take a cool look at the relative merits of the EPV and QGPV inversion systems and to examine some of the problems associated with them.

In sections 2 and 3 we briefly described the two systems. The EPV system is more accurate than the QGPV system, but it is more complicated, more difficult to understand, and less well defined. The main complicating feature of the EPV system is its nonlinearity: this poses difficulties of practical numerics, possible non-uniqueness of solutions, and non-superposability of solutions obtained from spatial decompositions of EPV. The "attribution" of flow and thermodynamic fields to particular spatial decompositions of EPV ("anomalies") is essential to the use of PV inversion as a technique to improve physical understanding of atmospheric flow. Attribution is much easier, and more justifiable, in the QGPV system because of its simplicity and linearity - but the expense in terms of reduced accuracy must always be borne in mind.

The lack of definition of the EPV system has become clear partly as a result of attempts to simplify the system, but ultimately it reflects the fact that the system is not related to any well-established consistent *prognostic* model of atmospheric behaviour. (The emergence of balanced prognostic models based on EPV conservation rather than energy conservation is a development to be followed with interest.) QGPV inversion has the advantage of a close relationship to a classical prognostic model of atmospheric flow.

In section 4 we discussed at length an important feature of both the EPV and the QGPV inversion systems: the freedom of choice for spatial boundary conditions. This feature reduces the definition of both systems. Following Hakim *et al.* (1996) we have described the QGPV inversion process as *static QGPV inversion*, and have contrasted it with *prognostic QGPV inversion*, which is the familiar process whereby streamfunction tendency is evaluated during the time integration of a classical quasi-geostrophic model. The boundary conditions

for *prognostic* QGPV inversion are unambiguously defined by non-controversial, inescapable physical requirements such as no normal flow through rigid boundaries. Some combinations of boundary conditions are not permitted in *static* QGPV inversion, and we have interpreted this feature via an isomorphism to a simple problem of heat diffusion, but wide ranges of boundary conditions *are* permitted. A similar situation holds with regard to the EPV inversion system, but is more difficult to investigate analytically.

As an aside, we express the view that prognostic QGPV inversion (Hakim *et al.* 1996, Nielsen-Gammon and Lefevre 1996) deserves further attention as an analysis method; it is well-defined in terms of the boundary conditions to be applied, and it directly confronts the time-evolution problem that is of foremost interest in weather forecasting.

The freedom to choose spatial boundary conditions in both EPV and (static) QGPV inversion has important consequences. One of the main aims of PV inversion is to increase understanding of atmospheric behaviour, both as a scientific objective and for the diagnosis and improvement of numerical model behaviour. Since the results of PV inversion depend on the chosen boundary conditions, there is a danger that the development of intuition and understanding may be misguided by the application of different boundary conditions in different studies. At the very least, it is important to be aware that the possibility of different boundary conditions *exists*, so that misleading comparisons can be avoided. We leave the reader to decide how much credence should be placed in a technique which depends on a convention to define it rather than on unambiguous physical reasoning.

A different, and immediately practical, application of PV inversion is in the *modification* of meteorological fields. The context may be either data assimilation, in which fields are to be modified prior to a numerical integration because information about the PV distribution is available, or post-processing, in which output fields are to be modified to allow for suspected model error. Examples of the data assimilation application are the studies of Demirtas and Thorpe (1999) and Swarbrick (2001), while the technique described by Carroll (1997) is routinely used by Met Office forecasters. The uncertainty in the boundary conditions can be seen in this context as a definite advantage. It offers useful scope for experimentation, and also the valuable opportunity to include desired features that would not be permitted if other physical considerations had pre-empted the specification of boundary conditions.

Finally (in sections 5 and 6) we have considered static QGPV inversion in the presence of mountains. In the Appendix, a formulation of the prognostic QG system for a general vertical coordinate has been derived, but we have noted that - alas! - it is inappropriate for terrain-following vertical coordinates. For static PV inversion, a way forward from this difficulty involves exact transformation of QGPV for pressure coordinates (or height coordinates, or other non terrain-following coordinates) to a corresponding terrain-following vertical coordinate form, with application of boundary conditions at the terrain height. We have noted that this procedure would not be allowed in a *prognostic* QG model; boundary conditions have to be applied at the lowest coordinate surface of the model to preserve integral conservation properties in a time integration. This difficulty typifies the differences between static and prognostic PV inversion which have been discussed in other sections. It could be overcome by applying the boundary conditions (expressing the *effects* of the mountains) at the lowest coordinate surface of the model; but this would of course make the introduction of the terrain-following coordinate redundant.

APPENDIX A

A QG model for a general non-terrain-following coordinate

To derive a η-coordinate QG system, geopotential Φ and all thermodynamic variables are first represented as deviations from reference-state profiles that are functions only of η:

$$F = F_0(\eta) + F'(x,\, y,\, \eta,\, t), \tag{A.1}$$

where $F = \Phi,\, \alpha,\, p,\, \theta,\, T...$ It is assumed that the deviations and their vertical gradients are small in the sense that $\left|F'\right| \ll |F_0|$ and $\left|\partial F'/\partial\eta\right| \ll |dF_0/d\eta|$. As noted in section 5, these assumptions are not sustainable if η is a terrain-following coordinate and mountains are present, but in other cases they may be justified by QG scale analysis similar to that given by Charney (1948), Charney and Stern (1962), Pedlosky (1964) and White (1977). Our derivation follows most closely that given by White (1977), which is valid for motion characterized by $Ro \ll 1$ and $RiRo \gg 1$. Here $Ro = U/f_0L$ is a Rossby number and $Ri = N^2H^2/U^2$ a Richardson number, H and L being vertical and horizontal space scales of the motion, and U a scale of horizontal flow variation. The quantity N^2H/g, which is often assumed to be small (Charney and Stern 1962, Pedlosky 1964), is permitted to be of order unity. The approximation

$$\frac{D}{Dt} \approx \left(\frac{\partial}{\partial t} + \mathbf{v}_g \cdot \nabla_\eta\right) \tag{A.2}$$

(in which ageostrophic and vertical advection are neglected) is made when the operand is not a reference state field. We call (A.2) the *QG approximation of* D/Dt. It is assumed that the meridional scale of variation L_y obeys $\beta L_y \ll f_0$, so that a $\beta-$plane treatment suffices.

(a) Hydrostatic balance

Transformation from z to η of the hydrostatic relation is straightforward. Since $\Phi = gz$,

$$g + \alpha\frac{\partial p}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial\Phi}{\partial\eta} + \alpha\frac{\partial p}{\partial\eta} = 0. \tag{A.3}$$

Hence, for the reference state,

$$\frac{d\Phi_0}{d\eta} + \alpha_0\frac{dp_0}{\partial\eta} = 0, \tag{A.4}$$

and for the deviation quantities (upon neglecting a second order term),

$$\frac{\partial\Phi'}{\partial\eta} + \alpha_0\frac{\partial p'}{\partial\eta} + \alpha'\frac{\partial p_0}{\partial\eta} = 0. \tag{A.5}$$

(b) Horizontal momentum equation

In terms of the horizontal gradient operator on η surfaces, ∇_η, the horizontal momentum equation may be written (assuming hydrostatic balance) as:

$$\frac{D\mathbf{v}}{Dt} + f\mathbf{k}\times\mathbf{v} + \nabla_\eta\Phi + \alpha\nabla_\eta p = 0 \tag{A.6}$$

(see Kasahara (1974)). Expand the material derivative, introduce $f = f_0 + \beta y$, and note that $\nabla_\eta F_0 = 0$ for any reference state profile; with $\dot{\eta} \equiv D\eta/Dt$, (A.6) becomes

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla_\eta + \dot{\eta}\frac{\partial}{\partial \eta}\right)\mathbf{v} + (f_0 + \beta y)\,\mathbf{k}\times\mathbf{v} + \nabla_\eta\left(\Phi^{'} + \alpha_0 p^{'}\right) + \alpha^{'}\nabla_\eta p^{'} = 0. \tag{A.7}$$

Define a geostrophic wind $\mathbf{v}_g$ and an associated streamfunction $\psi = (\Phi' + \alpha_0 p')/f_0$ (as in (5.1)), and an extended ageostrophic wind $\hat{\mathbf{v}}_{ag}$ by (5.2); (A.7) can then be written as

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla_\eta + \dot{\eta}\frac{\partial}{\partial \eta}\right)\mathbf{v} + f_0\mathbf{k}\times\hat{\mathbf{v}}_{ag} + \beta y\mathbf{k}\times\mathbf{v} = 0. \tag{A.8}$$

QG approximation of the material derivative is now applied, and $\mathbf{v}$ is replaced by $\mathbf{v}_g$ as the advected flow and in the term involving βy. In terms of $\nabla_\eta\psi$ $(= -\mathbf{k}\times\mathbf{v}_g)$, (A.8) becomes simply

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_g\cdot\nabla_\eta\right)\nabla_\eta\psi + + f_0\hat{\mathbf{v}}_{ag} + \beta y\mathbf{v}_g = 0. \tag{A.9}$$

(c) Vorticity equation

Taking the horizontal divergence $(\nabla_\eta\cdot)$ of (A.9) gives, without further approximation, the η-coordinate QG (vertical) vorticity equation:

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla_\eta\right)\left\{\nabla_\eta^2\psi + \beta y\right\} + f_0\nabla_\eta\cdot\hat{\mathbf{v}}_{ag} = 0. \tag{A.10}$$

(d) Thermodynamic relations and the streamfunction

The potential temperature θ obeys the relation

$$\ln\theta = \frac{1}{\gamma}\ln p + \ln\alpha + \text{constant}. \tag{A.11}$$

(Here $\gamma = c_p/c_v$, the ratio of principal specific heats.) Thus, for the reference state,

$$\frac{1}{\theta_0}\frac{d\theta_0}{d\eta} = \frac{1}{\gamma p_0}\frac{dp_0}{d\eta} + \frac{1}{\alpha_0}\frac{d\alpha_0}{d\eta}\,, \tag{A.12}$$

and for the deviation quantities (to first order)

$$\frac{\theta^{'}}{\theta_0} = \frac{p^{'}}{\gamma p_0} + \frac{\alpha^{'}}{\alpha_0}\,. \tag{A.13}$$

The geostrophic streamfunction $\psi = (\Phi' + \alpha_0 p')/f_0$ (see (5.1)) is the central prognostic variable of this QG system. Its vertical derivative can be related to the thermodynamic fields via (A.5):

$$f_0\frac{\partial\psi}{\partial\eta} = p^{'}\frac{\partial\alpha_0}{\partial\eta} - \alpha^{'}\frac{\partial p_0}{\partial\eta}\,. \tag{A.14}$$

A useful alternative form is obtained by using (A.13) to eliminate $\alpha^{'}$ in favour of $\theta^{'}$:

$$f_0\frac{\partial\psi}{\partial\eta} = -\alpha_0\frac{\partial p_0}{\partial\eta}\frac{\theta^{'}}{\theta_0} + S_0 p^{'}\,. \tag{A.15}$$

Here S_0 is the static stability factor defined by (5.4).

(e) Thermodynamic equation

The adiabatic form $D\theta/Dt = 0$ may be written in terms of $\theta^{'}$ and θ_0 as

$$\frac{D\theta^{'}}{Dt} + \dot{\eta}\frac{d\theta_0}{d\eta} = 0. \tag{A.16}$$

QG approximation of the material derivative allows (A.16) to be cast in the form

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_g \cdot \nabla_\eta\right)\frac{\theta^{'}}{\theta_0} + \frac{\dot{\eta}}{\theta_0}\frac{d\theta_0}{d\eta} = 0. \tag{A.17}$$

Upon use of (A.15), (5.3) and (5.4), (A.17) becomes

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_g \cdot \nabla_\eta\right)\frac{\partial\psi}{\partial\eta} - \frac{S_0}{f_0}\frac{dp_0}{d\eta}\widehat{w} = 0. \tag{A.18}$$

(f) Continuity equation

The η-coordinate continuity equation (with the hydrostatic approximation) is

$$\frac{D}{Dt}\left(\frac{\partial p}{\partial\eta}\right) + \frac{\partial p}{\partial\eta}\left(\nabla_\eta \cdot \mathbf{v} + \frac{\partial\dot{\eta}}{\partial\eta}\right) = 0 \tag{A.19}$$

(Kasahara 1974). In terms of p_0 and $p^{'}$, (A.19) becomes

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla_\eta + \dot{\eta}\frac{\partial}{\partial\eta}\right)\frac{\partial p^{'}}{\partial\eta} + \dot{\eta}\frac{d^2p_0}{d\eta^2} + \left(\frac{dp_0}{d\eta} + \frac{\partial p^{'}}{\partial\eta}\right)\left(\nabla_\eta\cdot\mathbf{v} + \frac{\partial\dot{\eta}}{\partial\eta}\right) = 0. \tag{A.20}$$

QG approximation of the material derivative in (A.20), and neglect of $\partial p^{'}/\partial\eta$ in comparison with $dp_0/d\eta$, gives the η-coordinate QG continuity equation in the form

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_g\cdot\nabla_\eta\right)\frac{\partial p^{'}}{\partial\eta} + \frac{dp_0}{d\eta}\nabla_\eta\cdot\mathbf{v} + \frac{\partial}{\partial\eta}\left(\dot{\eta}\frac{dp_0}{d\eta}\right) = 0. \tag{A.21}$$

Differentiation of (5.3) w.r.t. η gives

$$\left(\frac{\partial}{\partial t} + \mathbf{v}_g\cdot\nabla_\eta\right)\frac{\partial p^{'}}{\partial\eta} + \frac{\partial}{\partial\eta}\left(\dot{\eta}\frac{dp_0}{d\eta}\right) = \frac{\partial}{\partial\eta}\left(\widehat{w}\frac{dp_0}{d\eta}\right) - \left(\frac{\partial\mathbf{v}_g}{\partial\eta}\cdot\nabla_\eta\right)p^{'}. \tag{A.22}$$

Since $\nabla_\eta\cdot(\mathbf{k}\times\nabla_\eta q) = 0$ for any scalar field q, it follows from (5.2) that

$$\frac{dp_0}{d\eta}\nabla_\eta\cdot\mathbf{v} = \frac{dp_0}{d\eta}\nabla_\eta\cdot\widehat{\mathbf{v}}_{ag} + \frac{1}{f_0}\left[\nabla_\eta\left(\alpha^{'}\frac{dp_0}{d\eta}\right)\right]\cdot\left(\mathbf{k}\times\nabla_\eta p^{'}\right). \tag{A.23}$$

Use of (A.14) for $\alpha^{'}dp_0/d\eta$ allows (A.23) to be written as

$$\frac{dp_0}{d\eta}\nabla_\eta\cdot\mathbf{v} = \frac{dp_0}{d\eta}\nabla_\eta\cdot\widehat{\mathbf{v}}_{ag} + \left(\nabla_\eta\frac{\partial\Psi}{\partial\eta}\right)\cdot\left(\mathbf{k}\times\nabla_\eta p^{'}\right). \tag{A.24}$$

Upon noting (5.1) and the vector identity $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})$, (A.24) becomes

$$\frac{dp_0}{d\eta}\nabla_\eta\cdot\mathbf{v}=\frac{dp_0}{d\eta}\nabla_\eta\cdot\widehat{\mathbf{v}}_{ag}+\left(\frac{\partial\mathbf{v}_g}{\partial\eta}\cdot\nabla_\eta\right)p'. \tag{A.25}$$

By (A.22) and (A.25), the η-coordinate QG continuity equation (A.21) condenses to

$$\frac{dp_0}{d\eta}\nabla_\eta\cdot\widehat{\mathbf{v}}_{ag}+\frac{\partial}{\partial\eta}\left(\widehat{w}\frac{dp_0}{d\eta}\right)=0. \tag{A.26}$$

The compactness of (A.26) vindicates the definitions (5.2) and (5.3) of extended ageostrophic flow $\widehat{\mathbf{v}}_{ag}$ and extended vertical velocity $\widehat{w}$.

(g) QGPV

From (A.10) and (A.26):

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla_\eta\right)\left\{\nabla_\eta^2\psi+\beta y\right\}=f_0\left(\frac{dp_0}{d\eta}\right)^{-1}\frac{\partial}{\partial\eta}\left(\widehat{w}\frac{dp_0}{d\eta}\right). \tag{A.27}$$

From (A.18):

$$\widehat{w}\frac{dp_0}{d\eta}=\left(\frac{\partial}{\partial t}+\mathbf{v}_g\cdot\nabla_\eta\right)f_0S_0^{-1}\frac{\partial\psi}{\partial\eta}. \tag{A.28}$$

Eliminating $\widehat{w}dp_0/d\eta$ between (A.27) and (A.28) gives the QGPV equation (5.5).

(h) Omega equation

Equations (A.9) and (A.18) involve the local time derivatives of $\partial\psi/\partial x$, $\partial\psi/\partial y$ and $\partial\psi/\partial\eta$. Eliminating them by appropriate cross differentiation gives three diagnostic equations. For example, eliminating $\partial/\partial t\,(\partial^2\psi/\partial x\partial\eta)$ gives

$$f_0\frac{\partial\widehat{u}_{ag}}{\partial z}+(f_0S_0)^{-1}\frac{dp_0}{d\eta}\frac{\partial\widehat{w}}{\partial x}=2Q_x-\beta y\frac{\partial u_g}{\partial\eta}\,, \tag{A.29}$$

and eliminating $\partial/\partial t\,(\partial^2\psi/\partial y\partial\eta)$ gives

$$f_0\frac{\partial\widehat{v}_{ag}}{\partial z}+(f_0S_0)^{-1}\frac{dp_0}{d\eta}\frac{\partial\widehat{w}}{\partial y}=2Q_y-\beta y\frac{\partial v_g}{\partial\eta}\,. \tag{A.30}$$

Here Q_x and Q_y are the components of the η-coordinate Q-vector, as defined by (5.7). Elimination of $\widehat{u}_{ag}$ and $\widehat{v}_{ag}$ from (A.29) and (A.30) by use of (A.26) gives the η-coordinate omega equation (5.6). We observe that

$$\omega\equiv\frac{Dp}{Dt}=\frac{Dp'}{Dt}+\dot\eta\frac{dp_0}{d\eta}\,. \tag{A.31}$$

QG approximation of the material derivative on the right-hand side of (A.31) gives

$$\omega\approx\left(\frac{\partial}{\partial t}+\mathbf{v}_g\cdot\nabla_\eta\right)p'+\dot\eta\frac{dp_0}{d\eta}=\frac{dp_0}{d\eta}\widehat{w} \tag{A.32}$$

(see (5.3)). Hence the quantity $\widehat{w}dp_0/d\eta$ is a QG approximation of $\omega\equiv Dp/Dt$, so (5.6) can fairly be said to be the *omega* equation of the η-coordinate QG system.

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